Invariant elliptic curves as attractors in the projective plane

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Abstract

Let f be a rational self-map of $\mathbb{P}^2$ which leaves invariant an elliptic curve $\mathcal{C}$ with strictly negative transverse Lyapunov exponent. We show that $\mathcal{C}$ is an attractor, i.e. it possesses a dense orbit and its basin is of strictly positive measure.

Key words: Attractor, Lyapunov exponent.
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1 Introduction

Let f be a rational self-map of $\mathbb{P}^2$ of algebraic degree $d \geq 2$ which leaves invariant an elliptic curve $\mathcal{C}$ (i.e. an algebraic curve of genus one). We assume that $\mathcal{C}$ does not contain indeterminacy points. In [BDM], Bonifant, Dabija and Milnor study such maps and give several examples. They associate to f, a canonical ergodic measure $\mu_{\mathcal{C}}$, supported on $\mathcal{C}$, which possesses a positive Lyapunov exponent $\chi_1 = (\log d)/2$ in the tangent direction of $\mathcal{C}$. The transverse exponent corresponds to the second Lyapunov exponent χ_2 of $\mu_{\mathcal{C}}$, see section 3 for the definition.

An invariant compact set $A = f(A)$ will be called *an attractor* if A possesses a dense orbit and if the basin of A defined by

$$B(A) = \{x \in \mathbb{P}^2 \,|\, d(f^n(x), A) \to 0 \text{ as } n \to \infty\},$$

has strictly positive Lebesgue measure. Here, $d(.,.)$ denotes the distance in $\mathbb{P}^2$ with respect to a fixed Riemanian metric. The purpose of this article is to establish the following theorem which gives an affirmative answer to a conjecture of Bonifant, Dabija and Milnor.

Theorem 1.1 *Let f, $\mathcal{C}$ and $\mu_{\mathcal{C}}$ be as above. Assume that the transverse exponent χ_2 of $\mu_{\mathcal{C}}$ is strictly negative. Then $\mathcal{C}$ is an attractor.*

Under this assumption, $\mu_{\mathcal{C}}$ is a saddle measure, see [deT], [Di] for the construction of such measures in a similar context and [Si], [DS] for the basics on complex dynamics.

Recall that a rational self-map f of $\mathbb{P}^2$ of algebraic degree d is given in homogeneous coordinates $[z] = [z_0 : z_1 : z_2]$, by $f[z] = [F_0(z) : F_1(z) : F_2(z)]$ where F_0, F_1, F_2 are three homogeneous polynomials in z of degree d with no common factor. In the sequel, we always assume that $d \geq 2$. The common zeros in $\mathbb{P}^2$ of F_0, F_1, and F_2 form the indeterminacy set $I(f)$ which is finite. Let $\mathcal{C} \subset \mathbb{P}^2$ be an elliptic curve. Then, there exists a lattice Γ of $\mathbb{C}$ and a holomorphic map

$$\Psi : \mathbb{C}/\Gamma \to \mathbb{P}^2$$

such that $\Psi(\mathbb{C}/\Gamma) = \mathcal{C}$. Moreover, if S denotes the singular locus of $\mathcal{C}$, the map

$$\Psi : (\mathbb{C}/\Gamma) \setminus \Psi^{-1}(S) \to \mathcal{C} \setminus S$$

is a biholomorphism.

We say that $\mathcal{C}$ is *f-invariant* if $\mathcal{C} \cap I(f) = \varnothing$ and $f(\mathcal{C}) = \mathcal{C}$. In this case, the restriction $f_{|\mathcal{C}}$ lifts to a holomorphic self-map $\widetilde{f}$ of $\mathbb{C}/\Gamma$. Even if $\mathcal{C}$ is singular, f inherits several properties of $\widetilde{f}$. Like all holomorphic self-maps of $\mathbb{C}/\Gamma$, $\widetilde{f}$ is necessarily of the form $t \mapsto at + b$ and leaves invariant the normalized Lebesgue measure $\widetilde{\mu}_{\mathcal{C}}$ on $\mathbb{C}/\Gamma$. So, the topological degree of $\widetilde{f}$, i.e. the number of points in a fiber, is equal to $|a|^2$. It is not difficult to check that this degree is equal to d, see [BD]. Therefore, $|a|^2 = d \geq 2$. Then, by a classical theorem on ergodicity on compact abelian groups, $\widetilde{\mu}_{\mathcal{C}}$ is $\widetilde{f}$-ergodic, i.e. is extremal in the cone of invariant positive measures. Its push-forward $\mu_{\mathcal{C}}$ is an f-ergodic measure supported on $\mathcal{C}$. Moreover, generic orbits of $f_{|\mathcal{C}}$ are dense in $\mathcal{C}$. On the other hand, $f_{|\mathcal{C}}$ inherits of the repulsive behavior of $\widetilde{f}$ and $\mu_{\mathcal{C}}$ possesses a positive Lyapunov exponent equal to $\chi_1 = \log|a| = (\log d)/2$ in the tangent direction of $\mathcal{C}$. By Oseledec's theorem, see Section 3 below, we have

$$\chi_1 + \chi_2 = \frac{1}{2}\langle \mu_{\mathcal{C}}, \mathrm{Jac}(f)\rangle$$

where $\mathrm{Jac}(f)$ denotes the Jacobian of f with respect of the Lebesgue measure of $\mathbb{P}^2$. So, the hypothesis in Theorem 1.1 is equivalent to

$$\langle \mu_{\mathcal{C}}, \mathrm{Jac}(f)\rangle < \log d.$$

Some examples in [BDM] satisfy this condition and give the first attractors in $\mathbb{P}^2$ with non-open basins. The proof of our main result is based on

the study of the stable manifolds associated to $\mu_{\mathcal{C}}$. They are contained in the basin of $\mathcal{C}$. We show that their union has strictly positive measure. The ingredient is the use of quasi-conformal mappings and holomorphic motions.

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2 Quasi-conformal mapping and holomorphic motion

There are several equivalent definitions of quasi-conformal mappings (see [Ahl] for more details). Here, we give an analytic one.

Definition 2.1 *Let ϕ be a homeomorphism between two domains of $\mathbb{C}$. We say that ϕ is a quasi-conformal mapping if ϕ has locally integrable distributional derivatives which satisfy*

$$|\phi_{\overline{z}}| \leq k|\phi_z|$$

for some $k > 0$. A homeomorphism of $\mathbb{P}^1$ is a quasi-conformal mapping if it is locally quasi-conformal.

The following property of quasi-conformal mappings is crucial in our proof.

Proposition 2.2 *A quasi-conformal mapping sends sets of Lebesgue measure 0 to sets of Lebesgue measure 0.*

Now, we briefly introduce the notion of holomorphic motion. For a more complete account see [GJW]. For $r > 0$, we denote by Δ_r the disk centered at the origin in $\mathbb{C}$ with radius r and by Δ the unit disk. If E is a subset of $\mathbb{P}^1$, a holomorphic motion of E parametrized by Δ is a map

$$h : \Delta \times E \to \mathbb{P}^1$$

such that:

i) $h(0, z) = z$ for all $z \in E$,

ii) $\forall\, c \in \Delta$, $z \mapsto h(c, z)$ is injective,

iii) $\forall\, z \in E$, $c \mapsto h(c, z)$ is holomorphic on Δ.

By the works of Mañé, Sad, Sullivan, Thurston and Slodkowski (see [MSS], [ST] and [Slo]), any holomorphic motion h of E can be extended to a holomorphic motion $\widetilde{h}$ of $\mathbb{P}^1$. Furthermore, even if no continuity in z is assumed, $\widetilde{h}$ is continuous on $\Delta \times \mathbb{P}^1$. More precisely, we have the following result.

Theorem 2.3 *Let h be a holomorphic motion of a set $E \subset \mathbb{P}^1$ parametrized by Δ. Then there is a continuous holomorphic motion $\widetilde{h} : \Delta \times \mathbb{P}^1 \to \mathbb{P}^1$ which extends h. Moreover, for any fixed $c \in \Delta$, $\widetilde{h}(c, .) : \mathbb{P}^1 \to \mathbb{P}^1$ is a quasi-conformal homeomorphism.*

We shall need the following Lemma in the proof of Theorem 1.1.

Lemma 2.4 *Let h be a holomorphic motion of a Borel set $E \subset \mathbb{P}^1$ of strictly positive measure. Then $\cup_{c\in\Delta}\{c\} \times h(c, E)$ has strictly positive measure in $\Delta \times \mathbb{P}^1$.*

Proof. By Theorem 2.3, h can be extended to a holomorphic motion $\widetilde{h}$: $\Delta \times \mathbb{P}^1 \to \mathbb{P}^1$ such that, for any fixed $c \in \Delta$, $\widetilde{h}(c, .) : \mathbb{P}^1 \to \mathbb{P}^1$ is a quasi-conformal homeomorphism. Since a quasi-conformal mapping sends sets of measure 0 to sets of measure 0, it follows that for every $c \in \Delta$, we have $\mathrm{Leb}(h(c, E)) > 0$. So, by Fubini's theorem, $\cup_{c\in\Delta}\{c\} \times h(c, E)$ has strictly positive measure. $\square$

3 Hyperbolic dynamics

Suppose that g is a holomorphic self-map of a complex manifold M of dimension m. The following Oseledec's multiplicative ergodic theorem (cf. [KH] and [Wa]) gives information on the growth rate of $\|D_x g^n(v)\|$, $v \in T_x M$ as $n \to +\infty$. Here, $D_x g^n$ denotes the differential of g^n at x. Oseledec's theorem holds also when g is only defined in a neighbourhood of $\mathrm{supp}(\nu)$.

Theorem 3.1 *Let g be as above and let ν be an ergodic probability with compact support in M. Assume that $\log^+ Jac(g)$ is in $L^1(\nu)$. Then there exist integers k, $m_1, \ldots, m_k$, real numbers $\lambda_1 > \cdots > \lambda_k$ (λ_k may be $-\infty$) and a subset $\Lambda \subset M$ such that $g(\Lambda) = \Lambda$, $\nu(\Lambda) = 1$ and for each $x \in \Lambda$, $T_x M$ admits a measurable splitting*

$$T_x M = \bigoplus_{i=1}^{k} E_x^i$$

such that $\dim_{\mathbb{C}}(E_x^i) = m_i$, $D_x g(E_x^i) \subset E_{g(x)}^i$ *and*

$$\lim_{n\to+\infty} \frac{1}{n} \log \|D_x g^n(v)\| = \lambda_i$$

locally uniformly on $v \in E^i_x \setminus \{0\}$. Moreover, for $S \subset N := \{1, ..., k\}$ and $E^S_x = \oplus_{i \in S} E^i_x$, the angle between $E^S_{g^n(x)}$ and $E^{N \setminus S}_{g^n(x)}$ satisfies

$$\lim_{n \to +\infty} \frac{1}{n} \log \sin |\angle(E^S_{g^n(x)}, E^{N \setminus S}_{g^n(x)})| = 0.$$

The constants λ_i are the *Lyapunov exponents* of g with respect to ν. It is not difficult to deduce that

$$2 \sum_{i=1}^{k} m_i \lambda_i = \int \log \operatorname{Jac}(g) d\nu.$$

If all Lyapunov exponents are non-zero, we say that ν is *hyperbolic*. In this case, let $\lambda > 0$ such that $\lambda < |\lambda_i|$ for all $1 \leq i \leq k$ and let

$$E^s_x = \bigoplus_{\lambda_i < 0} E^i_x, \;\; E^u_x = \bigoplus_{\lambda_i > 0} E^i_x.$$

Then, for each point x in Λ and $\delta > 0$ we define the *stable manifolds* at x by

$$W^s_\delta(x) = \{y \in M \mid d(g^n(x), g^n(y)) < \delta e^{-\lambda n} \;\; \forall n \geq 0\}.$$

From Pesin's theory, we have the following fundamental result, see [BP], [PS], [RS] and [Pol] for more details.

Theorem 3.2 *Let $x \in \Lambda$. If $\delta > 0$ is small enough, then*

i) $W^s_\delta(x)$ is an immersed manifold in M,

ii) $T_x W^s_\delta(x) = E^s_x$.

4 Basin of an attracting curve

Since the support of $\mu_{\mathcal{C}}$ does not contain indeterminacy point, we have $\log^+ \operatorname{Jac}(f) \in L^1(\mu_{\mathcal{C}})$. We assume that $\mu_{\mathcal{C}}$ has a strictly negative transverse exponent χ_2. Then, there exists a hyperbolic set $\Lambda \subset \mathcal{C}$ such that $\mu_{\mathcal{C}}(\Lambda) = 1$ and $E^u_x = T_x\mathcal{C}$ for all $x \in \Lambda$.

The first step of the proof of Theorem 1.1 is to find, for some $p \in \Lambda$, an open neighbourhood where the stable manifolds are paire-wise disjoint. For this end, we prove that the restriction $f_{|\mathcal{C}}$ inherits of the repulsive behavior of $\widetilde{f}$. Recall that $d(.,.)$ is the distance on $\mathbb{P}^2$ and denote by $\widetilde{d}(.,.)$ the standard distance on $\mathbb{C}/\Gamma$.

Lemma 4.1 *There is a constant $\beta > 0$ such that for each $p \in \mathcal{C} \setminus S$ we can find $\alpha > 0$ with the property that, if $x, y \in \mathcal{C}$ are distinct points in the ball $B(p, \alpha)$ of radius α centered at p, then $d(f^n(x), f^n(y)) > \beta$ for some $n \geq 0$.*

Proof. As Ψ is one-to-one except on finitely many points, we can find a finite open covering $\{U_i\}_{i \in I}$ of $\mathbb{C}/\Gamma$ such that Ψ is injective on each $\overline{U_i}$. Let $z_1, z_2 \in \mathbb{C}/\Gamma$. We denote by $\epsilon > 0$ a Lebesgue number of this covering, i.e. if $\widetilde{d}(z_1, z_2) < \epsilon$ then, there exists $i \in I$ such that z_1 and z_2 are in U_i. Recall that one can choose $r > 0$ such that if $\widetilde{d}(z_1, z_2) < r$ then $\widetilde{d}(\widetilde{f}(z_1), \widetilde{f}(z_2)) = |a|\widetilde{d}(z_1, z_2)$. We can assume that $\epsilon < r$. Let $\widetilde{\alpha} > 0$ such that $2|a|\widetilde{\alpha} \leq \epsilon$. If $0 < \widetilde{d}(z_1, z_2) < \widetilde{\alpha}$ then there exists $n \geq 0$ such that

$$\widetilde{\alpha} < \widetilde{d}(\widetilde{f}^n(z_1), \widetilde{f}^n(z_2)) \leq |a|\widetilde{\alpha}.$$

Therefore, we can find $i \in I$ such that $\widetilde{f}^n(z_1)$ and $\widetilde{f}^n(z_2)$ are in U_i. So

$$d(f^n(\Psi(z_1)), f^n(\Psi(z_2))) > \beta$$

where

$$\beta = \min_{i \in I} \inf_{\substack{x_1, x_2 \in U_i \\ \widetilde{d}(x_1, x_2) > \widetilde{\alpha}}} d(\Psi(x_1), \Psi(x_2)) > 0.$$

Finally, for each $p \in \mathcal{C} \setminus S$ we can choose $\alpha > 0$ such that if $x, y \in B(p, \alpha) \cap \mathcal{C}$, there are preimages $\widetilde{x}$ and $\widetilde{y}$ of x and y by Ψ which satisfy $\widetilde{d}(\widetilde{x}, \widetilde{y}) < \widetilde{\alpha}$. Then, $d(f^n(x), f^n(y)) > \beta$ for some $n \geq 0$. $\square$

Lemma 4.2 *Let $p \in \Lambda$. There exist $\delta_0 > 0$ and an open neighbourhood U of p such that if $\delta < \delta_0$, $x, y \in U \cap \Lambda$, $x \neq y$ then $W^s_\delta(x) \cap W^s_\delta(y) = \varnothing$.*

Proof. By Lemma 4.1, we can choose for U the ball of radius α centered at p and $\delta_0 \leq \beta/2$. If there exist $x, y \in U \cap \Lambda$, $x \neq y$, with $W^s_\delta(x) \cap W^s_\delta(y) \neq \varnothing$, then for $z \in W^s_\delta(x) \cap W^s_\delta(y)$,

$$d(f^n(x), f^n(y)) \leq d(f^n(x), f^n(z)) + d(f^n(z), f^n(y)) \leq 2\delta e^{-\lambda n} \leq 2\delta,$$

for every n, which contradics Lemma 4.1. $\square$

Proof of Theorem 1.1. Let $p \in \Lambda$ be a regular point of $\mathcal{C}$. We choose suitable local coordinates (U, φ) at p, such that there exists an open set V in $\mathbb{C}$ with $\varphi(U) = \Delta \times V$ and $\varphi(U \cap \mathcal{C}) = \{0\} \times V$. Denote by π_1 and π_2 the projections of $\Delta \times V$ on Δ and V respectively. Let $x \in \Lambda \cap U$. By the stable manifold theorem, there exists $\delta(x) > 0$ such that $W^s_{\delta(x)}(x) \cap U$ is an immersed manifold in U. So, there exists a holomorphic embedding $\rho_x : \Delta \to U$ with $\rho_x(0) = x$

and $\rho_x(\Delta) \subset W^s_{\delta(x)}(x) \cap U$. Using these stable manifolds, with a suitable choice of $\delta(x)$, we will construct a holomorphic motion.

First, by Lemma 4.2, possibly after replacing U by an open subset, we can choose $\delta(x) < \delta_0$ for all $x \in \Lambda \cap U$. The stable manifolds $W^s_{\delta(x)}(x)$ are then paire-wise disjoint.

Since $W^s_{\delta(x)}(x)$ is tangent to E^s_x in x, we may assume, by choosing $\delta(x)$ sufficiently small, that the map defined by

$$\begin{aligned}\rho^1_x :&\Delta \to \Delta\\ &c \mapsto \pi_1 \circ \varphi \circ \rho_x(c),\end{aligned}$$

is injective and therefore is a biholomorphism onto its image. Moreover $\rho^1_x(0) = 0$, so there exists $r(x) > 0$ such that $\Delta_{r(x)} \subset \rho^1_x(\Delta)$.

As $\mu_{\mathcal{C}}(\Lambda) = 1$, there exists $a > 0$ and a subset Λ_a of $\Lambda \cap U$ such that $\mu_{\mathcal{C}}(\Lambda_a) > 0$ and for all points x in Λ_a, $r(x) > a$. Define $E = \varphi(\Lambda_a)$.We have $\mathrm{Leb}(E) > 0$. Define also

$$\begin{aligned}h :\ &\Delta_a \times E \to V \subset \mathbb{C}\\ &(c, z) \mapsto \rho^2_{\varphi^{-1}(0,z)} \circ (\rho^1_{\varphi^{-1}(0,z)})^{-1}(c),\end{aligned}$$

where, for all x in Λ_a, $\rho^2_x = \pi_2 \circ \varphi \circ \rho_x$. Since the stable manifolds are pair-wise disjoint, the map $z \mapsto h(c, z)$ is injective for each $c \in \Delta_a$. For all $x \in \Lambda_a$, ρ^2_x and $(\rho^1_x)^{-1}$ are well-defined and holomorphic, so for every $z \in E$, $c \mapsto h(c, z)$ is holomorphic on Δ_a. Therefore, h is a holomorphic motion of E parametrized by Δ_a. As $W = \cup_{x\in\Lambda_a} W^s_{\delta(x)}(x)$ is included in the attracting basin of $\mathcal{C}$ and $\varphi(W) = \cup_{c\in\Delta_a}\{c\} \times h(c, E)$, by the Lemma 2.4, the basin of $\mathcal{C}$ has strictly positive measure. $\square$

Remark 4.3 *By Hurwitz's formula, if $\mathcal{C}$ is an invariant curve then $\mathcal{C}$ is rational or elliptic. If $\mathcal{C}$ is rational and $f_{|\mathcal{C}}$ is a Lattès map, i.e a map which is semi-conjugated to an endomorphism of a torus, then its equilibrium measure is absolutely continuous with the respect to the Lebesgue measure. We obtain in the same way that its basin has strictly positive measure.*

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